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THE THEORY OF LONGITUDINAL VIBRATIONS OF A CONICAL ELASTIC BODY IN AN ELASTIC MEDIUM

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Aim. To elaborate the theory of longitudinal vibrations of a solid elastic body with one fixed end in the elastic medium. The example of such a body may be found in a sugar beet root in soil, the latter being elastic medium. **Methods.** The principle of stationary action of Ostrogradsky-Hamilton and the Ritz method were applied in the work. **Results.** The Ritz method was applied to obtain the Ritz frequency equation for the oscillating process under investigation. The analytic expressions were defined to determine the first and second eigenfrequencies of vibration and the amplitude of constrained vibrations of any of its cross-sections. The values of the first and second eigenfrequencies of the elastic body under investigation with specific geometric and physical parameters were found. The dependency diagrams for the first and second eigenfrequencies on the coefficient of elastic contraction of soil as the elastic medium, and the dependency diagrams for the amplitude of constrained oscillations of the mentioned body on the coefficient c of elastic deformation of soil and the distance of the cross-section of the body from the conditional point of fixation were drawn. The dependency diagrams for the amplitude of constrained oscillations of the elastic body on the change in the amplitude and the frequency of perturbing force were obtained. **Conclusions**. The impossibility of resonance occurrence was substantiated as the frequency of the perturbing force cannot equal the frequency of eigenvibrations of the elastic body due to technological and technical reasons. It was proven that the breaking of the elastic body is impossible with longitudinal deformations due to the shortness of the amplitude of longitudinal vibrations of the mentioned body.

Keywords: solid elastic body, elastic medium, functional of Ostrogradsky-Hamilton, longitudinal vibrations, eigenforms, eigenfrequencies, Ritz method.

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INTRODUCTION

When harvesting the sugar beet we always tried to avoid those problems that could break the beet roots in the process of digging them from the soil, disrupt their outer surface or cause their total damage, as well. If mentioned problems persist then they are accompanied by a significant loss of the harvest.

In particular, during the excavation of beet root from the soil by vibrations we can, with a given degree of accuracy, represent it as a resilient rod with one end fixed in the elastic medium and the rod is exposed to vibrations that are arising from working tool. What is more and very significant, the soil surrounding the roots, is also an elastic medium.

Fundamental analytical study of transverse vibrations of the body of the root was performed and published in [1]. Here the sugar beet root crop was simulated as a body of conical shape with one point fixed at the bottom and that has elastic properties. In this case, the transverse vibrations of the root body are described by differential equation with partial derivatives of the fourth order. The solution of this equation made it possible to determine the natural frequencies of free transverse vibrations of a root crop body. Directly, the process of extracting sugar beet roots from the soil, in this paper, is investigated further via composition of additional equations of kinetostatics, which allowed, with a certain degree of accuracy, find the conditions for the complete extraction of the root from the soil.

However, from constructional and technological point of view, the extraction of beet root from the soil in the good quality level, with the usage of transverse vibrations, proved impracticable. And that has stimulated the use of devices that provide the transfer of vibrations through the beet roots in the longitudinal vertical plane.

Fundamentally new formulation of the theory of vibrating excavation of beet roots from the soil when applying perturbing forces, namely in a longitudinal vertical plane, was published in papers [2–4]. The case of free and forced transverse vibrations of the body of the root when matching directions of perturbing forces with the direction of translational motion of a vibrating digging up the working body is published in the works [5–8] and is of interest both from theoretical and practical points of view.

To consider the more general problem about longitudinal vibrations of a continuous elastic conical body, fixed in an elastic medium.

MATERIALS AND METHODS

The problem is solved on the basis presented in publications [9–12], means, on the general theory of vibrations of straight rods with variable cross section. Also was used the methodology of research of mechanical systems, that is widely presented in [13, 14].

RESULTS AND DISCUSSION

We will consider such case when vibrational motion to the said body will be exerted in a longitudinal vertical plane. This position will corresponds to a situation in which vibrational forces of the harvesting machine will be applied on the both sides of the root (which is initially in the undisturbed soil) and during its extraction from the soil.

For the study of oscillations of holonomic systems with infinite number of degrees of freedom apply the principle of stationary action of Ostrogradskij-Hamilton [9]. In the theory of longitudinal, torsional and transverse vibrations of straight rods are used functionals Ostrogradsky-Hamilton, which in the most general form looks like (see [9]):

$$S = \iint_{t_0 0}^{t_1 l} L\left(t, x, y, \frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}, \frac{\partial^2 y}{\partial t^2}, \frac{\partial^2 y}{\partial t \partial x}, \frac{\partial^2 y}{\partial x^2}\right) dx dt, \qquad (1)$$

where $L = T - \Pi$ means Lagrange function; T is the kinetic energy of the system; Π is the potential energy of the system.

Using the principle of Ostrogradsky-Hamilton, explore the longitudinal vibrations of an elastic solid body, occurring under the action of vertical perturbing force, which varies according to a harmonic law, such as the following:

$$Q_{df.}(t) = H \sin \omega t, \qquad (2)$$

where H is the amplitude of the perturbing force; ω is the frequency of the perturbing force.

As we can see from the composed equivalent scheme (Fig. 1), the continuous elastic body is a crop root that has a conical shape (the angle at the vertex equal to 2γ, and the upper part is slightly higher than the soil surface), is modeled as a rod of variable cross-section with a fixed bottom end (point O). At the center of gravity, which is indicated by the point C, is applied force $\overline{G}$– body weight. Total length of the body is h. The connection of the body (the crop root) with the soil is determined by the total soil reaction $\overline{R_x}$, which is located down along the x-axis.

The above mentioned perturbing force $\overline{Q_{df.}}$ is applied to the body directly from its two sides, so in the diagram it is represented by two components $\overline{Q_1}$ and $\overline{Q_2}$. These forces are applied at a distance from the origin (point O) and they cause vibrations of the body (the root) in a longitudinal vertical plane, which disrupts its connection with the soil and create conditions of the extraction.

We form the Ostrogradskij-Hamilton's functional S for the vibrational process, which is being explored. With this aim we will introduce the necessary notations:

$F(x)$ – the area of the cross-section of the body at any point, which is at a distance x from the lower end (m^2);

E – Young's modulus for the material of the body (N/m^2); $y(x, t)$ – longitudinal displacement of any cross-section of the body at the time t (m); $Q(x, t)$ – the intensity of the external longitudinal load directed along the axis of the body (N/m); $\mu(x)$ – momentum of the body (kg/m).

According to [9] the functional of Ostrogradskij-Hamilton for longitudinal vibrations of straight bars is as follows:

$$S = \frac{1}{2}\int_{t_1}^{t_2}\int_{0}^{h}\left[\mu(x)\left(\frac{\partial y}{\partial t}\right)^2 - EF(x)\left(\frac{\partial y}{\partial x}\right)^2 + Q(x,\, t)\right]dxdt. \qquad (3)$$

We will find the expression of all the quantities in the functional (3). Considering that the body has a conical shape, we find that its cross-sectional area at a point which is located at any distance x from the point O, is equal to:

$$F(x) = \pi x^2 \mathrm{tg}^2\gamma. \qquad (4)$$

Evidently, the momentum of the body can be determined using following expression

$$\mu(x) = \rho\cdot\pi x^2 \mathrm{tg}^2\gamma, \qquad (5)$$

where ρ is the density of the body (kg/m^3).

Since the quantity $Q(x,\ t)$ which is the part of the functional (3), is the intensity of the distributed load (measured in (N/m)), the perturbing force should have a dimension of the intensity of the load. Using the first-order impulsive function $\sigma_1(x)$ [9] we can determine the intensity of a concentrated load, and thus incorporate concentrated forces in the load (distributed along the length).

If $Q_{df.}(t)$ is concentrated perturbing force applied at the point x_1 and is measured in N (Newton), then the function:

$$Q_{df.}(x,\, t) = Q_{df.}(t)\cdot\ \sigma_1(x - x_1), \qquad (6)$$

is measured in (N/m) and expresses the intensity of the concentrated load at the point x_1.

Function $\sigma_1(x - x_1)$ will be equal zero for all x, except $x = x_1$, where it goes to infinity.

If the perturbing force varies according to equation (2), then in accordance with the equation (6) can be written:

$$Q_{df.}(x,\, t) = H\sin\omega t\cdot\ \sigma_1(x - x_1). \qquad (7)$$

Since continuous elastic body is connected with the soil, which is also an elastic medium, then during the action of disturbing force (2) arises the force of soil resistance against displacement of the body which is doing vibrations. This force also affects the process of natural vibrations of the body in the soil, especially in the beginning of the oscillatory process, while body's connections with the soil are not broken yet.

It is obvious that the power of resistance of the soil (for the whole body) represents a distributed load on the area of the contact of the body with the soil, and therefore we can determine its intensity as the force of

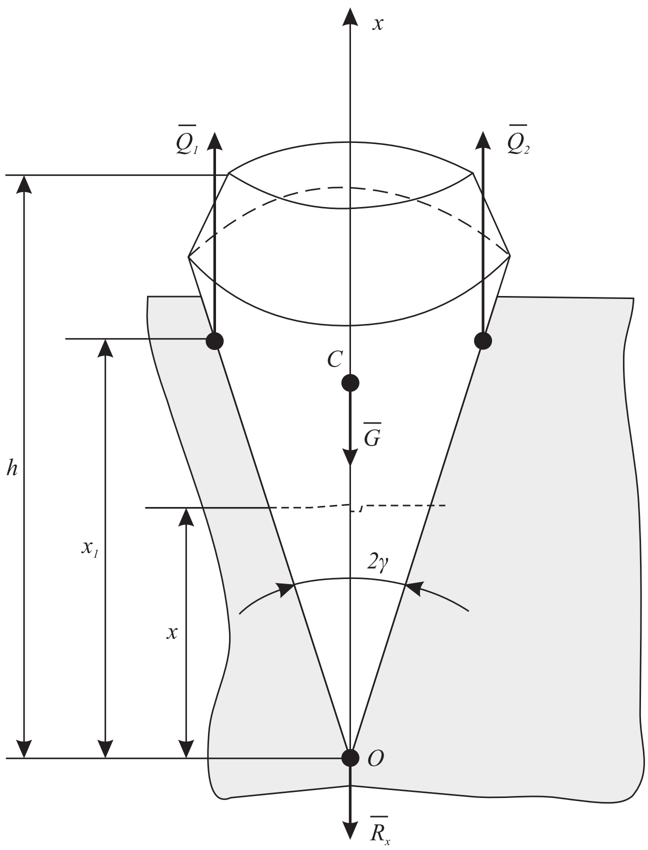

Fig. 1. The equivalent schema of longitudinal vibrations of the cone-shaped elastic body in an elastic medium

the soil resistance against the displacement of a unit length of the body.

Let c is the coefficient of the elastic soil deformation (ratio of the first coefficient of Winkler to the area of contact (N/m^3). Then the intensity $P(x,\ t)$ of the resistance of the soil against the body displacement at a point x will be equal to:

$$P(x,\, t) = 2\pi cx\cdot\mathrm{tg}\gamma\cdot y(x,\, t). \qquad (8)$$

Thus, we will have this relation for the longitudinal external load:

$$Q(x,\, t) = Q_{df.}(x,\, t) - P(x,\, t).$$

Considering expressions (4), (5), (7) and (8), the functional of Ostrogradskij-Hamilton (3) will have the form:

$$S = \frac{1}{2}\int_{t_1}^{t_2}\int_{0}^{h}\left\{\rho\cdot\pi x^2\mathrm{tg}^2\gamma\left(\frac{\partial y}{\partial t}\right)^2 - E\pi x^2\cdot\mathrm{tg}^2\gamma\left(\frac{\partial y}{\partial x}\right)^2 + \right.$$
$$\left. + H\sin\omega t\cdot\sigma_1(x - x_1)\cdot y(x,\, t) - 2\pi cx\cdot\mathrm{tg}\gamma\cdot y^2(x,t)\right\}dxdt. \qquad (9)$$

To find the eigenform and frequencies of longitudinal vibrations of the body in the soil we apply the method of Ritz [9]. According to this method, we will search harmonic longitudinal vibrations of the body in this form:

$$y(x,\, t) = \varphi(x)\sin(pt + \alpha), \tag{10}$$

where $\varphi(x)$ is eigenform of major vibrations; p is eigenfrequency of major vibration.

Because eigenforms and eigenfrequencies are associated with the free vibrations of the system, it is necessary to identify in the functional (9) that part which describes exactly the free vibrations of the system. It is clear that it will be the functional in following form:

$$S = \frac{1}{2}\int\limits_{t_1}^{t_2}\int\limits_{0}^{h}\Big[\rho\cdot\pi x^2 \mathrm{tg}^2\gamma\left(\frac{\partial y}{\partial t}\right)^2 - E\pi x^2\cdot\mathrm{tg}^2\gamma\left(\frac{\partial y}{\partial x}\right)^2 - \\ - 2\pi cx\cdot\mathrm{tg}\gamma\cdot y^2(x,t)\Big]\, dxdt. \tag{11}$$

Substituting expression (10) in the functional (11), we obtain:

$$S = \frac{1}{2}\int\limits_{t_1}^{t_2}\int\limits_{0}^{h}\Big\{\rho\cdot\pi x^2\cdot\mathrm{tg}^2\gamma\cdot\varphi^2(x)\cdot p^2\cdot\cos^2(pt+\alpha) - E\pi x^2\times \\ \times\, \mathrm{tg}^2\gamma[\varphi'(x)]^2\sin^2(pt+\alpha) - 2\pi cx\cdot\mathrm{tg}\gamma\cdot\varphi^2(x)\cdot\sin^2(pt+\alpha)\Big\}\, dxdt. \tag{12}$$

Integrating the expression (12) with respect to t within one period, we have:

$$S = \frac{\pi}{2p}\int\limits_{0}^{h}\Big\{\rho\pi x^2\mathrm{tg}^2\gamma\cdot\varphi^2(x)p^2 - E\pi x^2\mathrm{tg}^2\gamma[\varphi'(x)]^2 - \\ - 2\pi cx\cdot\mathrm{tg}\gamma\cdot\varphi^2(x)\Big\}\, dx. \tag{13}$$

According to the method of Ritz, values of the functional (13) are considered on a set of linear combinations of functions, *i. e.* expressions of the following form:

$$\varphi(x) = \sum_{i=1}^{n}\alpha_i\cdot\psi_i(x), \tag{14}$$

where α_i are parameters corresponding to definition; $\psi_i(x)$ − basis functions, which are specially selected and are known, they satisfy the geometric boundary conditions of the problem.

Thus, by substituting the expression (14) into expression (13), after appropriate adjustments we obtain:

$$S = \frac{\pi}{2p}\int\limits_{0}^{h}\Big[\rho\pi x^2\cdot\mathrm{tg}^2\gamma\cdot p^2\sum_{i,k=1}^{n}\psi_i(x)\psi_k(x)\alpha_i\cdot\alpha_k - E\pi x^2\cdot\mathrm{tg}^2\gamma\times \\ \times\sum_{i,k=1}^{n}\psi'_i(x)\psi'_k(x)\alpha_i\cdot\alpha_k - 2\pi cx\cdot\mathrm{tg}\gamma\sum_{i,k=1}^{n}\psi_i(x)\psi_k(x)\alpha_i\cdot\alpha_k\Big]\, dx. \tag{15}$$

Next, we introduce the following notations:

$$\begin{gathered}\int\limits_0^h \rho\cdot\pi x^2\cdot\mathrm{tg}^2\gamma\cdot\psi_i(x)\cdot\psi_k(x)dx = T_{ik}, \\ \int\limits_0^h E\pi x^2\cdot\mathrm{tg}^2\gamma\cdot\psi'_i(x)\cdot\psi'_k(x)dx = U_{ik}, \\ \int\limits_0^h 2\pi cx\cdot\mathrm{tg}\gamma\cdot\psi_i(x)\cdot\psi_k(x)dx = C_{ik}, \\ (i,k = 1,\, 2,\, ...,\, n).\end{gathered} \tag{16}$$

Substituting (16) into (15) gives the functional in the form of the function with parameters $\alpha_1, \alpha_2, ... \alpha_n$:

$$S_1(\alpha_1,\, \alpha_2,\, ...,\, \alpha_n) = \frac{\pi}{2p}p^2\sum_{i,\,k=1}^{n}T_{ik}\alpha_i\alpha_k - \frac{\pi}{2p}\sum_{i,\,k=1}^{n}U_{ik}\alpha_i\alpha_k - \\ - \frac{\pi}{2p}\sum_{i,\,k=1}^{n}C_{ik}\alpha_i\alpha_k. \tag{17}$$

We investigate the extreme of the functional (17). To do this, we will differentiate the expression (17) according to parameters α_i ($i = 1, 2, ..., n$) and obtained partial derivatives should be equal to zero. As a result, we obtain the system of linear homogeneous equations with the unknowns $\alpha_1, \alpha_2, ... \alpha_n$, from which (in turn) we find the equation for the frequencies of Ritz for longitudinal vibrations of a continuous elastic body, fixed in the soil:

$$\begin{vmatrix} p^2T_{11}{-}U_{11}{-}C_{11} & p^2T_{12}{-}U_{12}{-}C_{12} & \dots & p^2T_{1n}{-}U_{1n}{-}C_{1n} \\ p^2T_{21}{-}U_{21}{-}C_{21} & p^2T_{22}{-}U_{22}{-}C_{22} & \dots & p^2T_{2n}{-}U_{2n}{-}C_{2n} \\ \dots & \dots & \dots & \dots \\ p^2T_{n1}{-}U_{n1}{-}C_{n1} & p^2T_{n2}{-}U_{n2}{-}C_{n2} & \dots & p^2T_{nn}{-}U_{nn}{-}C_{nn} \end{vmatrix} = 0. \tag{18}$$

In practice, usually only the lower frequencies are determined, mostly the first and second one, which significantly affect the considered technological process.

To determine the first (main) frequency of natural vibrations, the equation (18) takes the following form:

$$p^2{}_1T_{11} - U_{11} - C_{11} = 0. \tag{19}$$

As a result of the solution of equation (19) we obtain an analytical expression for finding the first frequency:

$$p_1 = \sqrt{\frac{0.505\; E\cdot tg\gamma + 2.207\; ch}{0.917h\sqrt{\rho\cdot tg\gamma}}}\,. \tag{20}$$

For the calculation of the first eigenfrequency p_1 according to the expression (20) we take (in accordance with [15]) for the sugar beet h = 250 mm, γ = 14°, E = = 18.4·10⁶ N/m^2, ρ = 750 kg/m^3. According to [1] we take the coefficient c of the elastic deformation of the soil $c = 2 \cdot 10^5\, N/m^3$.

From the calculation we obtain: $p_1 = 496.4\ c^{-1}$ or $p_1 = 79$ Hz, which (with a high degree of accuracy) is consistent with the experimental data given in [16], according to which p_1 is in the range 75–120 Hz, and confirms similar data obtained theoretically in [17].

For the determining the first and second frequencies the equation (18) takes the form:

$$\begin{vmatrix} p^2T_{11}-U_{11}-C_{11} & p^2T_{12}-U_{12}-C_{12} \\ p^2T_{21}-U_{21}-C_{21} & p^2T_{22}-U_{22}-C_{22} \end{vmatrix} = 0. \tag{21}$$

Solving the equation (21) in the program Mathcad with the same parameters as in the previous case, there is obtained graphical dependence between the first, respectively the second natural frequency of the body of the root and the value of the coefficient of elastic deformation of the soil (see Fig. 2, 3).

As we can see from the graph (Fig. 2), with changing values of the coefficient c (coefficient of elastic deformation of the soil) within the range (0–2)×$\times 10^5\ N/m^3$, the values of the first angular frequency p_1 increases monotonically in the range 480–587 s^{-1}, or the frequencies are in the range 76.4–93.4 Hz.

From the graph on the Fig. 3 can be seen that with changing values of the coefficient c (coefficient of elastic deformation of the soil) within the range (0–2)×$\times 10^6\ N/m^3$, the second eigenfrequency of free vibrations varies in a small range: p_2 is from 3318 to 3344, or the frequencies are in the range 528–532 Hz.

Now we turn to the study of forced vibrations of a continuous elastic body. Purely forced vibrations will occur in accordance with the law:

$$y(x, t) = \varphi(x)\sin\omega t, \tag{22}$$

where $\varphi(x)$ is the form of forced vibrations.

To determine the form of forced oscillations of the body, we substitute the expression (22) in the functional (9), we obtain the following functional:

$$S_2 = \frac{1}{2}\int_{t_1}^{t_2}\int_0^h \Big\{ \rho\pi x^2 \cdot \mathrm{tg}^2\gamma \cdot \omega^2 \varphi^2(x)\cos^2\omega t - E\pi x^2 \mathrm{tg}^2\gamma[\varphi'(x)]^2 \sin^2\omega t + H\sigma_1(x-x_1)\varphi(x)\sin^2\omega t - 2\pi c x \cdot \mathrm{tg}\gamma \cdot \varphi^2(x) \cdot \sin^2\omega t \Big\} dx dt. \tag{23}$$

Integrating the expression (23) with respect to t within one period $T = 2\pi/\omega$, we have:

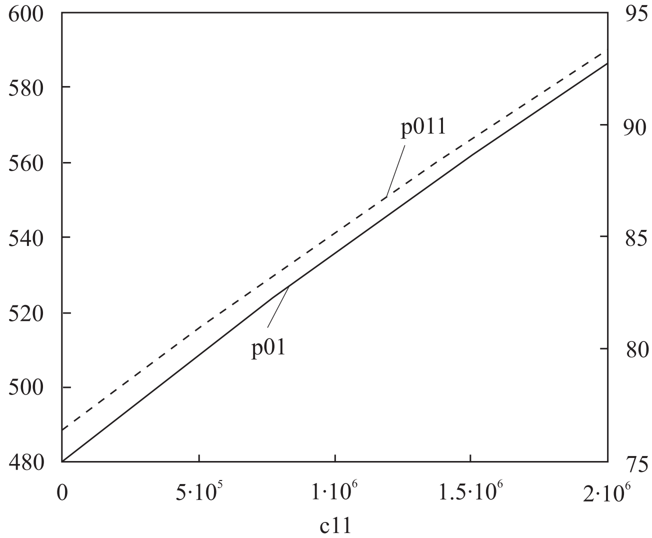

Fig. 2. The dependence between the first eigenfrequency of the longitudinal vibrations of the root and the coefficient c of elastic deformation of the soil (p01 – first angular frequency, p011 – first frequency, c11 – coefficient of elastic deformation)

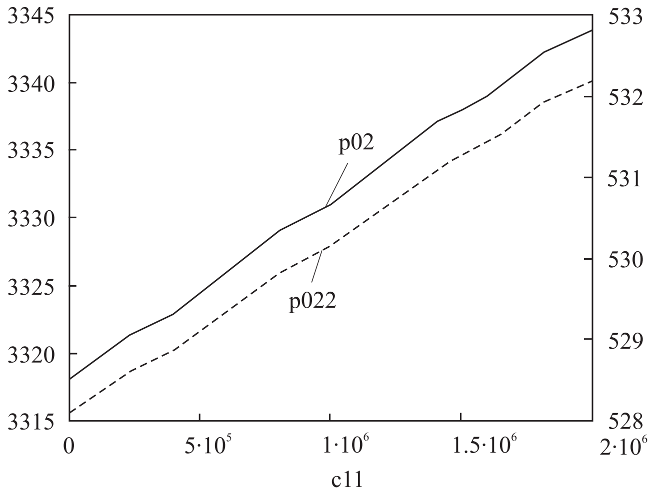

Fig. 3. The dependence between the second eigenfrequency of the longitudinal vibrations of the root and the coefficient c of elastic deformation of the soil (p02 – second angular frequency, p022 – second frequency, c11 – coefficient of elastic deformation)

$$S_2 = \frac{\pi}{2\omega}\int_0^h \Big\{ \rho\pi x^2 \mathrm{tg}^2\gamma \cdot \varphi^2(x)\omega^2 - E\pi x^2 \mathrm{tg}^2\gamma[\varphi'(x)]^2 + H\sigma_1(x-x_1)\varphi(x) - 2\pi c x \cdot \mathrm{tg}\gamma\varphi^2(x) \Big\} dx. \tag{24}$$

According to the method of Ritz, we consider the value of the functional (24) on the set of linear combinations of the following form:

$$\varphi(x) = \alpha\psi(x), \tag{25}$$

where α is the parameter corresponding to definition; $\psi(x)$ – basis function.

By substituting the expression (25) into functional (24) we obtain:

$$S_2 = \frac{\pi}{2\omega}\int_0^h\Big\{\rho\pi x^2 \mathrm{tg}^2\gamma\cdot\alpha^2\psi^2(x)\omega^2 - E\pi x^2\mathrm{tg}^2\gamma\cdot\alpha^2[\psi'(x)]^2 + \\ + H\sigma_1(x - x_1)\alpha\psi(x) - 2\pi cx\cdot\mathrm{tg}\gamma\cdot\alpha^2\psi^2(x)\Big\}\,dx. \quad (26)$$

We introduce the following notations:

$$\int_0^h \rho\cdot\pi x^2\cdot\mathrm{tg}^2\gamma\cdot\psi^2(x)dx = T, \quad (27)$$

$$\int_0^h E\pi x^2\cdot\mathrm{tg}^2\gamma\cdot[\psi'_i(x)]^2 dx = U, \quad (28)$$

$$\int_0^h 2\pi cx\cdot\mathrm{tg}\gamma\cdot\psi^2(x)dx = M, \quad (29)$$

$$\int_0^h \mathrm{H}\sigma_1(x - x_1)\cdot\psi(x)dx = L. \quad (30)$$

Substituting expressions (27)−(30) into (26) we obtain:

$$S_2 = \frac{\pi}{2\omega}\,[\omega^2 T\alpha^2 - (U + M)\alpha^2 + \mathrm{L}\alpha]. \quad (31)$$

Thus, on a set of functions (25) the functional (26) becomes a function with the independent variable α, which has the form (31).

The necessary condition for stationarity of the functional (31) (*i. e*. the existence of an extreme) is that the first derivative equals zero, namely:

$$\frac{\partial S_2}{\partial\alpha}\cdot\delta\alpha = 0, \quad (32)$$

which results in the following equation:

$$2\omega^2 T\alpha - 2(U + M)\alpha + L = 0, \quad (33)$$

from which we can find the required value of the parameter α. It will be:

$$\alpha = \frac{L}{2(U + M - \omega^2 T)}. \quad (34)$$

We take as the basis function $\psi(t)$ the form of forced longitudinal vibrations of the spike with constant cross-section with one end strongly fixed, which arise under the influence of the longitudinal harmonic force with frequency ω, applied at the point $x = x_1$.

According to [9], the form of forced vibrations of mentioned spike is as follows:

$$\psi(x) = D_1\sin ax \qquad \text{for } x \le x_1, \quad (35)$$

$$\psi(x) = D_2\cos a(h - x) \quad \text{for } x > x_1, \quad (36)$$

where

$$D_1 = -\frac{1}{aEF}\frac{\cos a(h - x_1)}{\cos ah}, \quad (37)$$

$$D_2 = -\frac{1}{aEF}\frac{\sin a\,x_1}{\cos ah}, \quad (38)$$

$$a = \omega\sqrt{\frac{\mu}{EF}}, \quad (39)$$

μ – momentum of the spike; F – cross-sectional area of the spike; E – Young's modulus for the material of the cone-shaped elastic body; h – length of the elastic conical body; ω – the frequency of forced vibrations of elastic conical body.

It is easy to check that the boundary conditions for basis functions (35) and (36) are satisfied and therefore, taken basis functions satisfy the requirements of the Ritz method.

To determine the parameter α we calculate the parameters T, U, M and L.

The result is:

$$T = \rho\pi\mathrm{tg}^2\gamma\Big\{D_1^2\Big(\frac{x_1^3}{6} - \frac{x_1^2\sin 2ax_1}{4a} - \frac{x_1\cos 2ax_1}{4a^2} + \\ + \frac{\sin 2ax_1}{8a^3}\Big) + D_2^2\Big[\frac{h^3}{6} - \frac{x_1^3}{6} + \frac{x_1^2\sin(2ah - 2ax_1)}{4a} + \quad (40) \\ + \frac{h}{4a^2} - \frac{x_1\cos(2ah - 2ax_1)}{4a^2} - \frac{\sin(2ah - 2ax_1)}{8a^3}\Big]\Big\},$$

$$U = E\pi\mathrm{tg}^2\gamma\Big[\frac{D_1^2 a^2 x_1^3}{6} + \frac{D_2^2 a^2(h^3 - x_1^3)}{6} + D_1^2\Big(\frac{x_1^2 a\sin 2ax_1}{4} + \\ + \frac{x_1\cos 2ax_1}{4} - \frac{\sin 2ax_1}{8a}\Big) - D_2^2\Big(\frac{x_1^2 a\sin(2ah - 2ax_1)}{4} + \\ + \frac{h}{4} - \frac{x_1\cos(2ah - 2ax_1)}{4} - \frac{\sin(2ah - 2ax_1)}{8a}\Big)\Big], \quad (41)$$

$$M = 2\pi cD_1^2\mathrm{tg}\gamma\Big(\frac{x_1^2}{4} - \frac{x_1\sin 2ax_1}{4a} + \frac{1 - \cos 2ax_1}{8a^2}\Big) + \\ + 2\pi cD_2^2\mathrm{tg}\gamma\Big[\frac{1 - \cos 2a(h - x_1)}{8a^2} - \frac{(h - x_1)^2}{4} + \frac{h(h - x_1)^2}{2} + \\ + \frac{x_1\sin 2a(h - x_1)}{4a}\Big], \quad (42)$$

$$L = HD_1\sin ax_1. \quad (43)$$

Substituting expressions (40)–(43) into the expression (34), we obtain the necessary value of the pa-

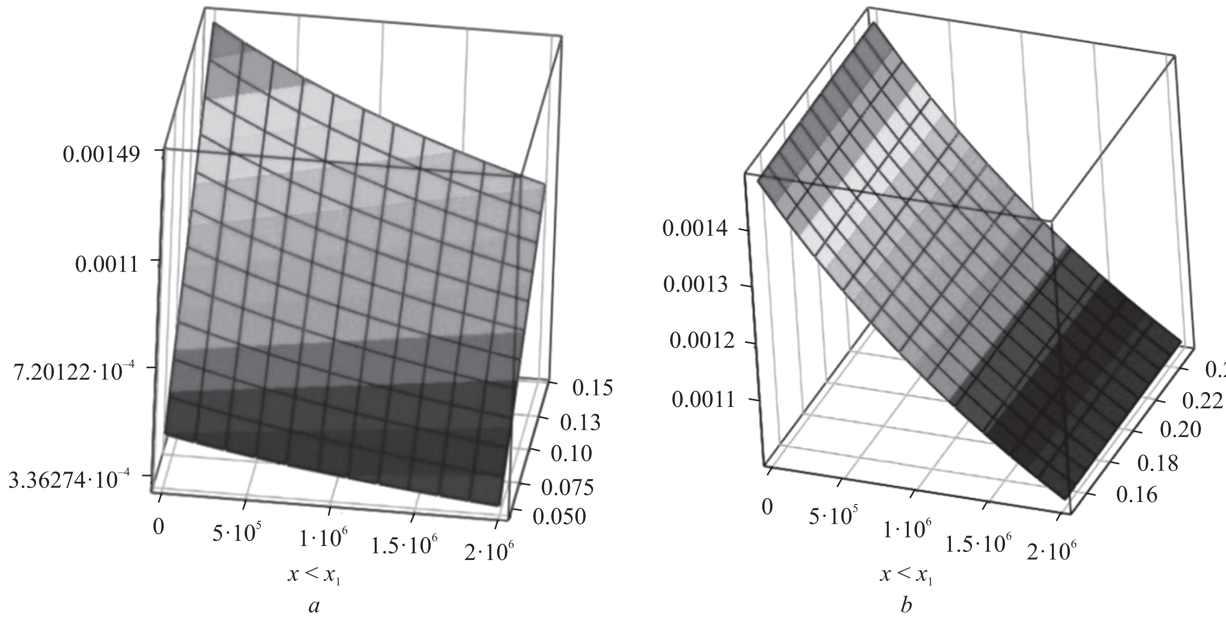

Fig. 4. The dependence of the amplitude of forced longitudinal vibrations of the conical body of root fixed in the soil in relation to the coefficient *c* (coefficient of elastic deformation of the surrounding soil) and to the distance *x* (distance between the cross-section from the conditional point of fixation x_1, Hz)

rameter α, whereby the functional (24) has a stationary value.

Considering the expressions (25), (35) and (36), we obtain expressions for the form of forced oscillations of a continuous elastic body fixed in the soil. They have the following form:

$$\varphi(x) = \alpha \cdot D_1 \sin ax, \text{ for } x \leq x_1,$$
$$\varphi(x) = \alpha \cdot D_2 \cos a(h - x), \text{ for } x > x_1, \quad (44)$$

where α is defined according to (34).

Substituting the expression (44) into (22), we finally obtain the law for forced vibrations of a continuous elastic body fixed in the soil:

$$y(x, t) = D_1\alpha \cdot \sin ax \cdot \sin \omega t, \text{ for } x \leq x_1,$$
$$y(x, t) = D_2\alpha \cdot \cos a(h - x) \cdot \sin \omega t, \text{ for } x > x_1. \quad (45)$$

On the basis of the results of theoretical investigations of forced vibrations of a continuous elastic body (root of sugar beet) fixed in the soil, we made the specific computation of the amplitude of these vibrations.

In the program Mathcad it was realized the calculation of the dependence of the amplitude of forced longitudinal vibrations of the conical body in relation to the *c* – coefficient of elastic deformation of the soil and to the distance of the cross-section of the root body from conditional point of fixation at the frequency of the perturbing force ν = 10 Hz and ν = 20 Hz and at the amplitude of this force $H = 500$ *N*.

According to the calculated results there were created graphs (Fig. 4).

As can be seen from the graphs, with increasing of the coefficient *c* (elastic deformation of the surrounding soil) the amplitude of forced vibrations of the root decreases, with increasing of the distance between the cross-section of root and the conditional point of fixation for $x < x_1$ the amplitude of the opposite − increases, while for $x > x_1$ − almost unchanged.

We calculated also the correlation between the amplitude of forced longitudinal vibrations of the mentioned elastic body and the amplitude of the perturbing force at the frequency ν = 20 Hz for the same parameters of the body, as above. The calculation was realized using the program Mathcad where the perturbing force was changing in the range 100–600 *N* for different cross-sections of the body. The graphs shown in Fig. 5 are the results of this calculation.

As can be seen from graphs, with an increase in the amplitude of the perturbing force the amplitude of forced longitudinal vibrations of the body of the root increases linearly. While, below the point of fixation ($x < 0.15$ m) with the increase in the distance of the

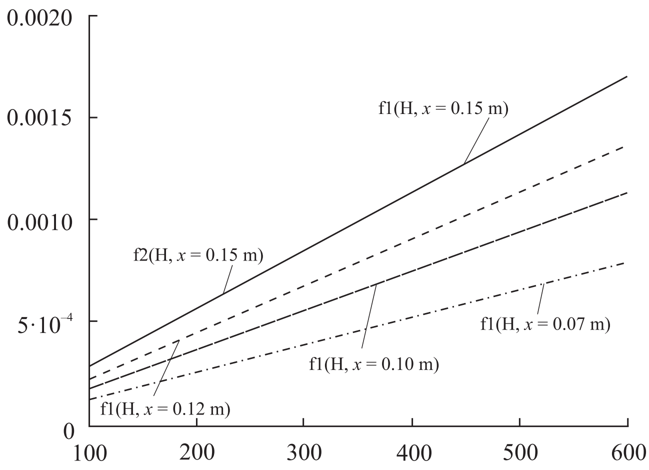

Fig. 5. Dependence between the amplitude of forced longitudinal vibrations of the elastic body of root and the amplitude of the perturbing force ($x < x_1$, $\nu = 20$ Hz)

root cross-section from the conditional point of fixation *O*, the amplitude also increases. Thus, for x = 0.07 he range of the amplitude is 0.13–0.8 mm; for x = = 0.1 m – in the range 0.19–1.14 mm; for x = = 0.12 m – in the range 0.23–1.36 mm; for x = = 0.15 m (point of fixation) – in the range 0.28–1.7 mm. However, above the point of fixation ($x > 0.15$ m) with the increase in the distance of the cross-section of the root from the conditional point of fixation *O*, the amplitude remains almost unchanged.

Because the first eigenfrequency of the considered example of a sugar beet as an elastic conical body is not less than 75 Hz, and the frequency of the perturbing force by for technological and technical reasons cannot be greater than 20 Hz, the resonance case is not possible, in fact. In addition, the calculated value of the amplitude of forced longitudinal vibrations of a root crop body, which is in the range 0.13−1.7 mm shows that the rupture of the root at its longitudinal deformation is also impossible.

CONCLUSIONS

On the basis of the use of the variation principle of Ostrogradskij-Hamilton we obtained equations for calculation of the natural frequencies of any order for longitudinal vibrations of a continuous elastic body with one fixed end. Thus, there were obtained the analytical expressions for finding the first and second eigenfrequency and also expressions for finding the amplitude of forced vibrations for any cross-section of a continuous elastic body with respect to its balance position. Given theoretical analyses open up opportunities for study the process of disruption of connections of crop roots with the soil during their harvesting by vibrational method.

Теорія поздовжніх коливань конусоподібного пружного тіла у пружному середовищі

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Мета. Створити теорію поздовжніх коливань суцільного пружного тіла у пружному середовищі з одним закріпленим кінцем. Прикладом такого тіла може бути розташований у ґрунті коренеплід цукрового буряка, причому ґрунт, у якому він знаходиться, також є пружним середовищем. **Методи.** Застосовано принцип стаціонарної дії Остроградського–Гамільтона і метод Ритца. **Результати.** За допомогою методу Ритца отримано рівняння частот Ритца для досліджуваного коливального процесу. Виписано аналітичні вирази для визначення першої і другої власних частот коливань тіла і амплітуди вимушених коливань будь-якого його поперечного перерізу. Знайдено величини першої і другої власних частот пружного тіла з конкретними геометричними і фізичними параметрами. Одержано графіки залежності першої і другої власних частот від коефіцієнта пружної деформації ґрунту як пружного середовища, а також графіки залежності амплітуди вимушених коливань зазначеного тіла від коефіцієнта *c* пружної деформації ґрунту і відстані поперечного перерізу тіла від умовної точки закріплення. Складено графіки залежності амплітуди вимушених коливань пружного тіла від зміни амплітуди і частоти збурювальної сили. **Висновки.** Обґрунтовано неможливість настання резонансу, оскільки частота збурювальної сили не може дорівнювати частоті власних коливань пружного тіла з технологічних і технічних причин. Доведено, що через малі розміри амплітуди поздовжніх коливань пружного тіла його розрив при поздовжніх деформаціях неможливий.

Ключові слова: суцільне пружне тіло, пружне середовище, функціонал Остроградського−Гамільтона, поздовжні коливання, власні форми і частоти, метод Ритца.

Теория продольных колебаний конусообразного упругого тела в упругой среде

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Цель. Создать теорию продольных колебаний сплошного упругого тела в упругой среде с одним закрепленным концом. Примером такого тела может быть расположенный в почве корнеплод сахарной свеклы, причем почва, в которой он находится, также является упругой средой. **Методы.** Применены принцип стационарного действия Остроградского–Гамильтона и метод Ритца. **Результаты.** При помощи метода Ритца получено уравнение частот Ритца для рассматриваемого колебательного процесса. Выписаны аналитические выражения для определения первой и второй собственных частот колебаний тела и амплитуды вынужденных колебаний любого его поперечного сечения. Найдены значения первой и второй собственных частот рассматриваемого упругого тела с конкретными геометрическими и физическими параметрами. Составлены графики зависимости первой и второй собственных частот от коэффициента упругой деформации почвы как упругой среды, а также графики зависимости амплитуды вынужденных колебаний указанного тела от коэффициента *c* упругой деформации почвы и расстояния поперечного сечения тела от условной точки закрепления. Получены графики зависимости амплитуды вынужденных колебаний упругого тела от изменения амплитуды и частоты возмущающей силы. **Выводы.** Обоснована невозможность наступления резонанса, поскольку частота возмущающей силы не может быть равна частоте собственных колебаний упругого тела по технологическим и техническим причинам. Доказано, что из-за малости амплитуды продольных колебаний упругого тела его разрыв при продольных деформациях невозможен.

Ключевые слова: сплошное упругое тело, упругая среда, функционал Остроградского−Гамильтона, продольные колебания, собственные формы и частоты, метод Ритца.

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